Proof of crossing formula for 2D percolation

1995 J. Phys. A: Math. Gen. 28 6479

(http://iopscience.iop.org/0305-4470/28/22/023)

J. Phys. A: Math. Gen. 28 (1995) 6479–6480. Printed in the UK

ADDENDUM

Proof of crossing formula for 2D percolation

Robert M Ziff

Department of Chemical Engineering, University of Michigan, Ann Arbor, MI 48109-2136, USA

Received 11 July 1995

Abstract. The author's recently conjectured expression for Cardy's crossing formula in 2D percolation is rewritten in terms of theta and elliptic functions, and verified explicitly. Exact results for aspect ratio r equal to integral powers of two are also given.

In [1], the author conjectured that Cardy's result [2] (see also [3]) for the crossing probability in percolation

$$\pi_v(r) = c\,\eta^{1/3}{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta) \tag{1}$$

where $\eta = (1-k)^2/(1+k)^2$, $r = 2K(k^2)/K(1-k^2)$ and $c = 3\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2$, can be written directly in terms of r as

$$\pi_v'(r) = -\frac{2^{4/3}\pi c}{3}\left[\sum_{n=-\infty}^{\infty}(-1)^n \mathrm{e}^{-3\pi r(n+1/6)^2}\right]^4 \tag{2}$$

where $\pi_v(r)$ is the probability density of crossing a rectangular system of height r and of unit width in the vertical direction, and $\pi_v'(r)$ (the derivative with respect to r) gives the probability density that the maximum height of clusters grown from the bottom of an infinitely high rectangular system is equal to r (assuming free boundaries on the sides in both cases). The form of (2) was conjectured from a series development and verified to high order, but not proven explicitly. In this addendum, I provide that proof, and also give alternative expressions for (2).

Those alternative expressions are

$$\pi_v'(r) = -\frac{2^{4/3}\pi c}{3}\mathrm{e}^{-\pi r/3}\prod_{n=1}^{\infty}(1-\mathrm{e}^{-2\pi r n})^4 \tag{3a}$$

$$= -\frac{\pi c}{3}\left(\vartheta_1'\right)^{4/3} \tag{3b}$$

$$= -\frac{4c}{3\pi}\left[\eta(1-\eta)\right]^{1/3} K^2(\eta)\,. \tag{3c}$$

The first result is implied by 24.2.1 of [4] or (13a,b) of [1], and the second puts this product in terms of the Jacobi theta function $\vartheta_1' = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})^3 = 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}$ where $q = \mathrm{e}^{-\pi r}$. The third expression follows by applying $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$ and formulae for ϑ_i in terms of the elliptic integral K. Note that in [1] it was shown that r and η are directly related according to $r = K(1-\eta)/K(\eta)$. Expanding (3a) or (3b) in powers of q yields the series expansion of $\pi_v'(r)$ given in [1].

Equation (3c) above leads directly to an explicit proof of (2). The derivative of Cardy's result (1) is given by [1]

$$\pi_v'(r) = -\frac{c}{3[\eta(1-\eta)]^{2/3}} \frac{K^2(\eta)}{\dot{K}(1-\eta)K(\eta) + K(1-\eta)\dot{K}(\eta)} \tag{4}$$

where the dot represents differentiation with respect to the argument (a prime is used to indicate the complementary argument $K'(\eta) = K(\eta_1) = K(1-\eta)$). This result is equivalent to (3c) if the relation

$$\dot{K}(1-\eta)K(\eta) + K(1-\eta)\dot{K}(\eta) = \frac{\pi}{4\eta(1-\eta)} \tag{5}$$

is valid. But this identity follows directly from $\dot{K} = (E - \eta_1 K)/2\eta\eta_1$ and Legendre's relation $EK' + E'K - KK' = \pi/2$ [5], and thus, the equivalence of (1) and (2) follows.

In [1] it was shown that Landen's transformation can be used to find how η scales with r: $\eta(2r) = \{(1-[1-\eta(r)]^{1/2})/(1+[1-\eta(r)]^{1/2})\}^2$. Applying this same transformation to (3c) yields

$$\pi_v'(2r) = \pi_v'(r)\left(\frac{\eta^2(r)}{256(1-\eta(r))}\right)^{1/6} \tag{6}$$

which also implies $[\pi_v'(2r)]^6 = [\pi_v'(4r)]^2[\pi_v'(r)]^4 + 16[\pi_v'(4r)]^4[\pi_v'(r)]^2$. These yield $\pi_v'(2)/\pi_v'(1) = 2^{-3/2}$, $\pi_v'(4)/\pi_v'(2) = 2^{-7/4}(2^{1/2}-1)$, etc. Furthermore, $\pi_v'(1) = \Gamma(\frac{1}{4})^4/(\Gamma(\frac{1}{3})^3 2^{5/3} 3^{1/2}\pi) \approx 0.520\,246\,1715$, so closed expressions for $\pi_v'(r)$ for all r equal to powers of two follow. (Note $\pi_v'(1/r) = r^2\pi_v'(r)$.) Finally, a plot of $\pi_v'(r)$ shows that its maximum is at $r \approx 0.523\,5217$ (where $(\mathrm{d}/\mathrm{d}q)\vartheta_1' = 0$) with value $\pi_v' \approx -0.737\,3222$.

Corrections to [1] are as follows: $k(4) = 2^{5/4}/(2^{1/2}+1) \approx 0.985\,171\,431$ and $\eta(4) = [(2^{1/4}-1)/(2^{1/4}+1)]^4$ on p 1253. Also, the series in (16) can be found to all orders directly by using $\eta = \vartheta_2^4/\vartheta_3^4$.

Acknowledgments

This material is based upon work supported by the US National Science Foundation under Grant No DMR-9520700.

References

[1] Ziff R M 1995 *J. Phys. A: Math. Gen.* **28** 1249
[2] Cardy J 1992 *J. Phys. A: Math. Gen.* **25** L201
[3] Langlands R P, Pouliot Ph and Saint-Aubin Y 1994 *Bull. Am. Math. Soc.* **30** 1
[4] Abramowitz M and Stegun I A (ed) 1964 *Handbook of Mathematical Functions* (Washington, DC: US Government Printing Office).
[5] Whittaker E T and Watson G N 1927 *Modern Analysis* (Cambridge: Cambridge University Press) 4th edn, p 521